Stochastic resonance in two-dimensional Landau Ginzburg equation

2004 J. Phys. A: Math. Gen. 37 L391

(http://iopscience.iop.org/0305-4470/37/32/L01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) L391–L398 PII: S0305-4470(04)79228-8

LETTER TO THE EDITOR

Stochastic resonance in two-dimensional Landau Ginzburg equation

R Benzi[1] **and A Sutera**[2]

[1] Dipartimento di Fisica and INFM, Univ. di Roma "Tor Vergata", via della Ricerca Scientifica 1, 00173, Roma, Italy
[2] Dipartimento di Fisica Univ. di Roma "La Sapienza", P.le Aldo Moro 4, 00185, Roma, Italy

E-mail: sutera@romatm9.phys.uniroma1.it

Received 7 April 2004
Published 28 July 2004
Online at stacks.iop.org/JPhysA/37/L391
doi:10.1088/0305-4470/37/32/L01

Abstract
We study the mechanism of stochastic resonance for the Landau Ginzburg equation in two space dimensions, perturbed by a white noise. We review how to renormalize the equation in order to avoid ultraviolet divergences. Next, we show that the renormalization amplifies the effect of the small periodic perturbation in the system. We argue that stochastic resonance can be used to highlight the effect of renormalization in a spatially extended system with multiple, stable statistical steady states.

PACS number: 05.40.−a

1. Introduction

About 20 years ago, in studying the possible causes of the ice ages (Sutera 1980, 1981, Nicolis and Nicolis 1981), we introduced the concept of stochastic resonance (Benzi *et al* 1981). This process (hereafter, SR) requires that the phase space of a physical system is not homogeneous and the time spent in different regions of the phase space is not uniformly distributed. If the system is stochastically forced, the probability of large fluctuations can be periodically locked to a small external periodic forcing.

Relevant implications of the SR are the generality of the mechanism and the limited band, in the parameter space, wherein the mechanism occurs. The latter allows for its exploitation for enhancing the signal-to-noise ratio in a noisy system, while the former makes the concept applicable to a large class of systems. Furthermore, we note that, for the mechanism to occur, the only requirement is that the system has large deviation properties. Therefore, the restriction to systems stochastically perturbed is just a case amenable to analytic evaluation of the resonance range in the parameter space.

At the time of the above-mentioned publication, we had doubts whether the mathematical process leading to stochastic resonance was a stochastic process at all. Fortunately in Friedlin (2000) it has been proved that there exists a wide class of stochastic process obeying to a SR mechanism. Therefore we feel confident that, at least in a finite-dimensional phase space, the SR is a well-posed mathematical problem. Recently, however, it has been noted that spatially extended fields (Niemela *et al* 2000, Ganopolski and Rahmstorf 2002) may possess unimodal or multimodal probability densities depending on whether their behaviour is measured at different time or spatial scales. In this paper, we wish to establish few relationships that show similar behaviour if we model the system as a stochastic process in more than one space dimension. In addition, we extend the SR mechanism for these kinds of systems.

Stochastically perturbed spatially extended systems may lead to a non-consistent mathematics since the equations for the backward and the forward probability density functions may not exist. Moreover, when a white noise in space and time is considered, the resulting equations are plagued by divergent correlation functions in more than one space dimension.

In a series of papers by Jona-Lasinio and Mitter (1985), Jona-Lasinio and Mitter (1990) and Jona-Lasinio and Seneor (1991), it was proved that these problems could be overcome by the method of the renormalization theory. On the basis of these rigorous mathematical results, in this paper we wish to study the consequences of applying an external periodic (in time) forcing to a simple stochastic differential equation in two space dimensions.

2. Renormalization of a Landau Ginzburg equation

As discussed in many papers, a nonlinear Landau Ginzburg equation describes many physical systems including the lowest order of complexity paleoclimate models. Keeping this in mind, we discuss some of the consequences of stochastically perturbing such an equation when an external, slowly varying forcing (such as it may be induced by the secular variation of the earth's orbital parameters) is applied to the system.

The equation is

$$\partial_t \phi_\Lambda = \nu\,\Delta\phi_\Lambda - \frac{\partial V(\phi_\Lambda)}{\partial\phi_\Lambda} + \epsilon\frac{\mathrm{d}W_\Lambda}{\mathrm{d}t} \tag{1}$$

where

$$V(\phi) = \tfrac{1}{2}m\phi^2 + \tfrac{1}{4}g\phi^4. \tag{2}$$

In (1) we have introduced the cut-off field ϕ_Λ, where Λ is an ultraviolet cut-off in the Fourier space. Namely, denoting $\Phi(\vec{k})$ as Fourier transform of $\phi(x)$ we have

$$\phi_\Lambda = \int_{|\vec{k}|<\Lambda} \Phi(\vec{k})\,\mathrm{e}^{\mathrm{i}\vec{k}\vec{x}}\,\mathrm{d}\vec{k} \tag{3}$$

and an analogous definition has been used for $\frac{\mathrm{d}W_\Lambda}{\mathrm{d}t}$, where $\frac{\mathrm{d}W}{\mathrm{d}t}$ is a white noise delta correlated both in space and in time.

The field ϕ is assumed to be real and defined on the torus $[0, L]X[0, L]$, $L = 2\pi$, with periodic boundary conditions. Here, the order parameter ϕ may be interpreted, for instance, as a bulk climate variable while the origin of the nonlinearity may be connected to the several feedback present in the system. Note that the previous equation, in general, describes mean-field theories nearby critical points, therefore, the use of white noise in space and time is mandatory if we do not want to introduce unknown physical scales. For $m > 0$ the theory has no broken symmetries, as we assume in the following. Thus, in the absence of stochastic forcing, the solution of (1) has a single steady state $\phi = 0$. If a periodic external forcing is acting no stochastic resonance will occur. Therefore, the results presented here considerably differ from other approaches leading to enhancement of SR response.

Just to give a physical flavour to our study, we mention that the phenomenon of SR has been widely studied in literature since it was introduced in a variety of applications (see Benzi *et al* (1982) and Nicolis (1982) for a climate theory framework). Gaimmaitoni *et al* (1998) also provide a review and several applications including the case of spatially extended field.

The study of SR for the Landau Ginzburg equation in one space dimension, with a broken symmetry, i.e. $m > 0$, has been presented in Benzi *et al* (1985). In this case, the stochastic equation does not show any particular novelty since the probability density has no peculiarity. The stochastic solution behaviour follows the structure of ordinary differential equations. As we shall see, the extension to more spatial dimension leads to interesting consequences on the nature of the solutions.

As has been discussed in Benzi *et al* (1989), (1) has no limit for $\Lambda \to \infty$ in more than one space dimension unless it is renormalized. Physically, this implies that the statistical properties of ϕ depend on the cut-off scale Λ^{-1}. To avoid this unwished feature, one needs to renormalize the equation. In $D = 2$, the renormalization amounts to change any nonlinear term in (1) by its Wick product, a rather well-known procedure in quantum field theory. By using the Wick product, (1) becomes

$$\partial_t \phi_\Lambda = \nu\, \Delta\phi_\Lambda m\phi_\Lambda - g\phi_\Lambda^3 + 3gE_m(\Lambda)\phi_\Lambda + \epsilon \frac{\mathrm{d}W_\Lambda}{\mathrm{d}t}. \tag{4}$$

In (4) the quantity $E_m(\Lambda)$ is the expectation value of the second-order moment of the field Z_Λ satisfying the linear stochastic differential equation:

$$\partial_t Z_\Lambda = \nu\, \Delta Z_\Lambda - mZ_\Lambda + \epsilon \frac{\mathrm{d}W_\Lambda}{\mathrm{d}t}. \tag{5}$$

In two space dimensions one has

$$E_m(\Lambda) = C\frac{\epsilon^2}{\nu} \ln\left[\frac{\nu\Lambda^2 + m}{m}\right] \tag{6}$$

where C is a constant. Finally, let us recall that the quantity $\phi_\Lambda^3 - 3E_m(\Lambda)\phi_\Lambda$ is the Wick product of ϕ_Λ^3. One can show that (4) has a well-defined limit for $\Lambda \to \infty$. Let us remark that this limit is achieved by keeping the finite volume constant. In this case, one can show that equation (1) becomes

$$\partial_t \phi_\Lambda = \nu\, \Delta\phi_\Lambda - \frac{\partial V_{\mathrm{eff}}(\phi_\Lambda)}{\partial\phi_\Lambda} + \epsilon \frac{\mathrm{d}W_\Lambda}{\mathrm{d}t} \tag{7}$$

where

$$V_{\mathrm{eff}}(\phi) = \tfrac{1}{2}\rho\phi^2 - \tfrac{1}{4}g\phi^4. \tag{8}$$

The value of ρ is determined by the following set of equations:

$$\rho = -m + 3g[E_m(\Lambda) - E_\alpha(\Lambda)] \tag{9}$$

$$\alpha = \frac{\partial^2 V_{\mathrm{eff}}}{\partial\phi^2} \qquad \text{at} \quad \phi = \phi_e \tag{10}$$

where ϕ_e are the equilibriums of the effective potential, i.e., ϕ_e is determined by solving the equation

$$\frac{\partial V_{\mathrm{eff}}}{\partial\phi} = 0. \tag{11}$$

The set of equations (9)–(11) have a clear physical meaning, shortly reviewed as follows. First, let us note that ρ is independent of Λ in the limit $\Lambda \to \infty$ because of (6). In (9) there is a redefinition of the Wick product in terms of a new constant α, which is defined in equation (10) as the curvature of the effective potential at one of its equilibrium values, namely ϕ_e. The physical idea of (10) is that the counter term, introduced to renormalize the Landau Ginzburg equation, may produce a double-well potential. In this case, the position of the new minimum, determined by ρ, must be consistent with the fluctuations around the minimum itself. Using (9)–(11) one gets a nonlinear equation for the variable $x = \frac{\alpha}{m}$, namely

$$x = -2 + 2K \log x \tag{12}$$

where $K = \frac{3gC\epsilon^2}{m\nu}$. Let us remark, for future purpose, that it is possible to show that a solution of (12) always satisfies $x > 2K$. As discussed in Benzi *et al* (1989), the effective potential describes rather well the statistical properties of the space average field of (1). In general, we can consider the variables $\Phi(l)$ defined as the average in space of $\phi(x)$ on a box of side l. For some $l < L = 2\pi$, the statistical properties of $\Phi(l)$ are no longer described by the same value of ρ because of the increasing effect, at small scales, of the stochastic fluctuations. In particular, for $l \to 0$, one finds that the statistical properties of $\Phi(l)$ are described by a single-well potential centred around $\Phi(l) = 0$.

3. Effects on stochastic resonance

We want now to discuss the solutions of (1) when a small periodic perturbation is added to the system and in particular we want to discuss the effect of renormalization of the mechanism on the stochastic resonance. To this end, following the original discussion given in Benzi *et al* (1981), we consider the case of a time-independent constant A added to the rhs of (1). The crucial point is the computation of the effective potential V_{eff} previously introduced. While (9) remains unchanged (10) and (11) become

$$\alpha = -\rho + 3g{\phi_e}^2 \tag{13}$$

$$\rho\phi_e - g{\phi_e}^3 + A = 0. \tag{14}$$

The effect of the constant perturbation is to change the equilibrium solution of the effective potential and therefore to change the fluctuations around the equilibrium solution, which in turn change the value of ρ. The latter effect is due only to renormalization. To understand qualitatively the contribution due to renormalization, one can compute a perturbative solution of the set of equations (9), (13) and (14) in power of A. At first order one obtains

$$\rho_{1,2} = \rho_0 + \frac{6g\phi_{e0}H}{1-2H}\frac{A}{2\rho_0} \tag{15}$$

$$\phi_{e1,e2} = \phi_{e0} + \frac{1+H}{1-2H}\frac{A}{2\rho_0} \tag{16}$$

where $H = \frac{3gC\epsilon^2}{\nu\alpha_0}$ ($\alpha_0 = 2\rho_0$) and all quantities with the index 0 refer to values computed for $A = 0$. The previous definition of H implies $H = \frac{K}{x} < \frac{1}{2}$ because of the discussion on the solution of equation (12). Let us remark that we obtain two different values of ρ and ϕ_e because two different values of ϕ_{e0} are possible, one negative and the other positive. Thus there exist two possible effective potentials, each describing the two wells, respectively. By using equations (15) and (16) it is now possible to compute the effective potential difference between the two stable solutions and the unstable solution. One gets

$$\Delta V_{1,2} = -\frac{\rho^2}{4g} + A_R\phi_{e0} \tag{17}$$

$$A_R = A\left(1 + \frac{3H}{2(1-2H)}\right) \tag{18}$$

In the limit of $\epsilon \to 0$, we have $H \to 0$ and the effect of renormalization disappears. For the finite value of the noise and for g large enough the renormalization amplifies the asymmetry of the double-well potential. Therefore, we should expect that, in the two-dimensional Landau Ginzburg equation, the mechanism of SR acts for smaller value of the forcing amplitude with respect to the one estimated by looking at the equilibrium probability distribution with no periodic forcing. Also, as we discuss later on, SR should disappear at small enough scales.

To test the renormalization effect on SR, we have performed a numerical simulation on square lattice of 16×16 points. The parameters of our numerical simulation are $g = 220, \nu = 0.1, m = 0.1, \epsilon = 0.1$. On the lattice, the quantity $E_m(\Lambda)$, defined in (6), is given by

$$E_m(a) = \Sigma_{p,q} \frac{a^2\epsilon^2}{2S(m, \nu, q, p)} \tag{19}$$

where

$$S(m, \nu, q, p) = ma^2 + 4\nu - 2\nu\ \cos(2\pi q/N) - 2\nu\ \cos(2\pi p/N).$$

In (19) we introduce the lattice mesh $a = 2\pi/N$. It follows that $\Lambda = a^{-1}$.

For different values of A, by applying the Newton method, we have computed the solution of (9), (13) and (14), using (19). It turns out that, with our numerical parameters, $\rho_0 = 4.9$ and $A_R \sim 2A$, i.e., the external forcing is amplified by almost a factor 2. We have performed a long time integration of (1) with $A = 0$. The equilibrium probability distribution of $\Phi(L)$ has been found to be bimodal with maxima centred at $\pm\sqrt{\frac{\rho_0}{g}}$ and the average transition time τ between the two statistical steady states $\tau = 9$ in time units of m^{-1}. Next, using the analytical theory of SR discussed in Benzi *et al* (1981), we have computed, by taking into account the amplification of the renormalization, the value of A for which the system should show a clear SR. We found $A = 0.1$. Finally, we have applied a periodic perturbation on the rhs of (1) with amplitude $A = 0.1$ and period equal to $2\tau = 18$, i.e., two times the average transition time. We have performed a long time integration corresponding to 2048 time units. In figure 1, we show the power spectrum of the average field $\Phi(L)$. As one can clearly see, there is an extremely well-defined peak at frequency 113 which corresponds to a period of 18 time units. In order to show the amplification effect on SR due to renormalization, we have numerically integrated the zero-dimensional stochastic differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \rho_0 x - gx^3 + A\ \cos\left(\frac{2t\pi}{18}\right) + \sigma\frac{\mathrm{d}W}{\mathrm{d}t} \tag{20}$$

where $A = 0.1$ as before and the variance σ of the white noise has been tuned in order to reproduce an average transition time of 9 ($\sigma = 0.0165$) for $A = 0$. With this choice of σ we have been able to reproduce the equilibrium probability distribution of $\Phi(L)$ at $A = 0$ for the 2D Landau Ginzburg equation. As for the previous case, we have integrated (20) for 2048 time units. In figure 2, we show the power spectrum of x. At variance with figure 1, only a small effect of the periodic forcing is felt by the system, i.e., the system does not exhibit any SR. The comparison between figures clearly indicates that the renormalization effect is acting as an amplifier of the external forcing, as previously discussed in the framework of a perturbation theory.

4. Spatial scale dependence

In this section we discuss what we feel to be the main result of this paper, namely, the scale dependence of stochastic resonance phenomenon for the Landau Ginzburg equation. This

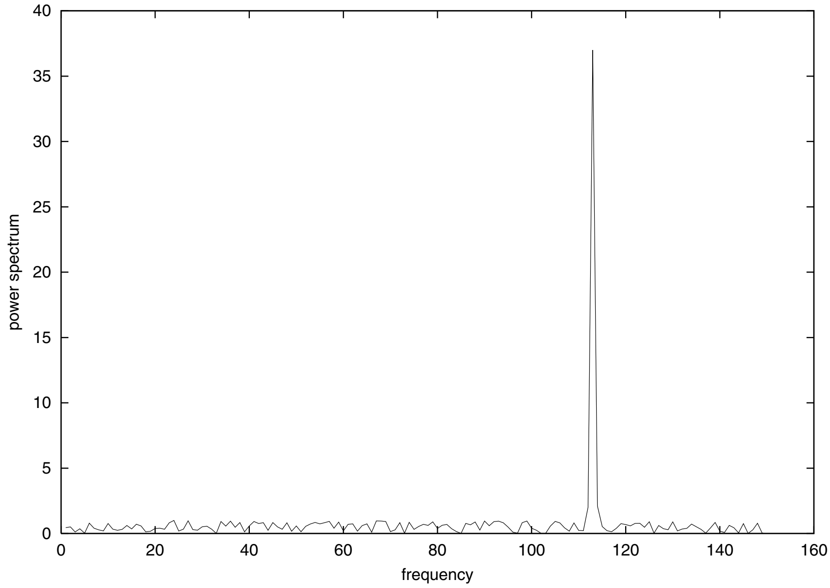

Figure 1. Power spectrum for the space average field $\Phi(L)$ obtained for the numerical simulation of (1). The peak corresponds to a period of 18 time units.

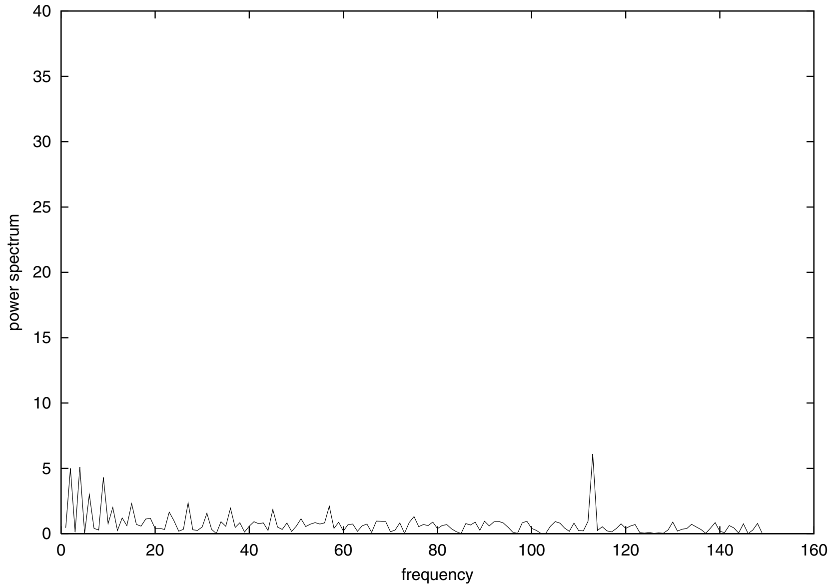

Figure 2. Same as in figure 1(*a*) for the zero-dimensional stochastic differential equation (20). Note that the amplitude of the peak at a period of 18 time units is much smaller with respect to figure 1.

property, together with spontaneous symmetry breaking, is applicable only to our approach, while stochastic resonance enhancement mechanism may be obtained under other assumptions either on the nature of the nonlinear dynamics or choosing special spatial non-white noise.

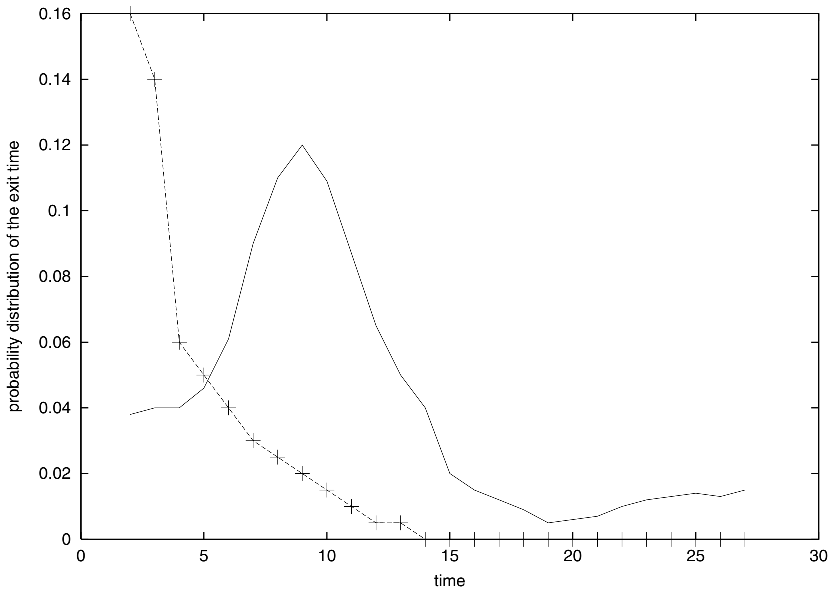

Figure 3. Probability distribution of the transition time for $\Phi(L)$ (continuous curve) and $\Phi(\frac{L}{16})$ (continuous curve with symbols).

For our purpose, it is quite interesting to compute the statistics of the transition times between the minima at different scales as described by the variables $\Phi(l)$. In the absence of forcing, the probability distribution of the transition time for $\Phi(L)$ is exponential. In the stochastic resonance, the probability distribution of the transition time for $\Phi(L)$ should show a well-defined maximum at 9 time units. On the other hand, if we consider the transition time for $\Phi(l)$ with $l \ll L$, then we expect that the forcing is not able to produce any SR. This is due to the fact that the parameter describing the effective potential for small l is changed and, in particular, the average transition time between the minima becomes much smaller (see Benzi *et al* (1989)). This effect is indeed observed in our numerical simulation. In figure 3, we show the probability distribution of the transition time for $\Phi(L)$ and $\Phi\left(\frac{L}{16}\right)$. As predicted, for the small scale, the probability distribution of the transition time is exponential. We can therefore reach the conclusion that, because of the renormalization, the SR mechanism becomes scale dependent—a feature that has not been previously observed.

5. Final remarks

In this paper, we have shown that renormalization can amplify the SR in the two-dimensional Landau Ginzburg equation and the enhancement is scale dependent. We want to remark that this effect can also be used to test, in a given experimental situation, whether or not a renormalization mechanism is acting in the system, as may be the case for the experimental results in Rayleigh–Benard turbulence discussed in Niemela *et al* (2000). In this case SR can be used as a tool to measure quantitatively the effect, if any, of renormalization in spatially extended systems. Moreover, we argue that the present effect may not be restricted to the stochastically perturbed system of gradient type and that, in principle, it could be detected in spatially extended deterministic systems.

Acknowledgment

We thank Professor Jona Lasinio for useful discussions.

References

Benzi R, Sutera A and Vulpiani A 1981 *J. Phys. A: Math. Gen.* **14** L453
Benzi R, Parisi G, Sutera A and Vulpiani A 1982 *Tellus* **34** 10
Benzi R, Sutera A and Vulpiani A 1985 *J. Phys. A: Math. Gen.* **18** 2239
Benzi R, Jona Lasinio G and Sutera A 1989 *J. Stat. Phys.* **55** 505
Friedlin M I 2000 *Physica* D **137** 333
Gammaitoni L, Hanngi P, Jung P and Marchesoni F 1998 *Rev. Mod. Phys.* **70** 223
Ganopolski A and Rahmstorf S 2002 Abrupt glacial climate changes due to stochastic resonance *Phys. Rev. Lett.* **8803** 038501
Jona-Lasinio G and Mitter P K 1985 *Commun. Math. Phys.* **101** 409
Jona-Lasinio G and Mitter P K 1990 *Commun. Math. Phys.* **130** 110
Jona-Lasinio G and Seneor R 1991 *J. Phys. A: Math. Gen.* **24** 4123
Nicolis C and Nicolis G 1981 *Tellus* **33** 225
Nicolis C 1982 *Tellus* **34** 1
Niemela J J, Skrbek L, Sreenivasan K R and Donnelly R J 2000 *Nature* **406** 439
Sutera A 1980 *J. Atmos. Sci.* **37** 245
Sutera A 1981 *Quart. J. R. Meteor. Soc.* **107** 137